Batting the Ball

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The velocity vector of a ball struck by a bat is a stated function of the ball and bat velocities, bat orientation, and certain constants. In the light of the equations of the collision, the operation and the consequences of swinging the bat are analyzed, and the role of the constants is discussed.

IT would be easy to list a dozen popular games in which the central event is a collision between a ball and a bat, club, mallet, racket, paddle, or stick of some sort. Frequently, the player's intention is to transfer to the ball as much momentum as possible, and always he has some problem of directional guidance. We investigate the mechanics of these processes with such generality and realism as may be conveniently introduced. The aim of the study is not to reform batting but to understand it.

The state of the bat at the moment of contact with the ball is defined by 13 independent variables, all of which are subject to the batter's control. These quantities are the 3 positional coordinates of the mass center (or other reference point) of the bat, 3 coordinates of angular orientation, 3 of linear momentum, 3 of angular momentum, and 1 coordinate of time. In his control of any one of these variables, the batter may err in either the positive or the negative sense, so it appears that he is faced at the outset with 26 roads to failure.

It should be said immediately that several of these errors matter so little that, individually, they do not cause failure; and among the others, there is wide variance in the need for precise determination as well as in the difficulties of control. Furthermore, the number which require exact management varies among the several sports. The baseball bat is supposed to be held with the trademark uppermost, but this is one of the simplest of the batter's responsibilities since it is an easy one to fulfill and since other rotational positions of the trademark have almost no effect upon performance unless they are implicated in the breaking of the bat. In tennis, however, the corresponding variable is of much importance since a small angular displacement may make the difference between a perfect placement and a netted ball.

The timing of his stroke makes no demands upon the skill of the golfer since his ball is there to be struck at any time, but the pitched baseball crosses the plate in about 10 msec and the batter must locate his swing in time with approximately this precision. The rotational position of the bat about the vertical axis is a critical matter in all the varieties of tennis, in hockey, and in golf; in baseball, this angle does not require such strict control except in such specialties as place hitting and bunting. There are few cases in which the angular velocities of the bat at the instant of contact are significant in themselves, though they may be practically involved in the production of important linear velocities.

In baseball, the vertical coordinate of the bat at contact is both important and hard to control. Most strike-outs result from its mismanagement, and the 1962 world championship was finally determined by an otherwise perfect swing of a bat which came to the collision 1 mm too high to effect the transfer of title.

STRESSES AND STRAINS

The elasticity of the collisions to be discussed ranges from a relative high in the golf drive, to a low in the return of a handball. Figure 1 shows stress–strain relations for the baseball and golf ball as recorded by an automatic materials testing machine. The lower loop represents the response of a golf ball which was pressed between the flat steel surfaces of the testing machine until its vertical thickness, initially the ball's diameter, was reduced by 0.38 in. during a continuous closure at the constant speed of 2 in./min. At the maximum displacement shown the machine was reversed and the ball allowed to regain its original figure. The narrow loop speaks for the high elasticity of the golf ball, which returned 0.78 of the energy expended in its compression. It should be noted that the work done is represented by the area to the *left* of the curve. High elasticity does not, of course, require Hooke's-law behavior; the type of curvature shown in all four branches is characteristic of the compression of any object which, like the sphere, enlists more and more of its substance to oppose the external force as the distortion proceeds.

The baseball (upper loop) was carried through a cycle like that of the golf ball except that it was pressed between two crossed sections of a regulation bat so that the ball might be cylindrically indented and not merely flattened. This arrangement was employed so that the strains might in some degree resemble those imposed in actual batting, but it must be recognized that while these curves tell us something about the basic properties of the spheres under study they are not directly applicable to the more interesting questions about brief impacts. In the first place, the batted or driven ball is deformed through external contact on *one side only*, and secondly, the internal stresses and yields in the case of the impulsive blow are certainly not duplicated in slow motion in any quasistatic distortion cycle. If a dependable similitude of this kind obtained we should be able to calculate from Fig. 1 the times of contact and coefficients of restitution effective in high power impacts, but it was found that times of contact so computed are too low for agreement with the actual ones, while calculated coefficients of restitution are too high. It appears that the mechanical efficiency of a sudden blow is less than that of a slow one which invests a like amount of work.

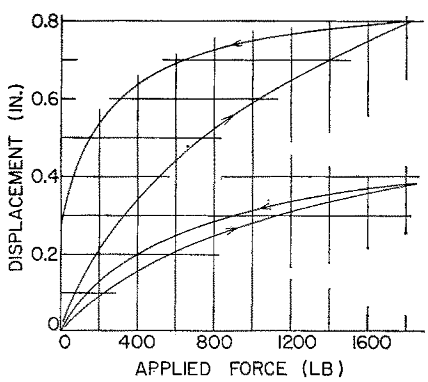

FIG. 1. Stress–strain cycles for the baseball (above) and golf ball.

EFFECT UPON THE BALL

The intention of the batter is to impart a velocity to the ball, often the maximum possible. The elongated body, here called a *bat*, is characterized by a mass M and a moment of inertia $I = Mk^2$ with respect to its mass center. The ball is a smooth sphere of mass m and of unspecified radius. The bat and ball have a mutual coefficient of restitution e which is assumed (somewhat erroneously) to be constant.

For ease of reading, this analysis is confined to motions in a horizontal plane; there is no basic difficulty about a three-dimensional derivation but the additional insight afforded would not be sufficient to justify the extra complication. We choose as x axis a horizontal line passing through the point of contact between ball and bat and disposed normally to the axis of the latter. The x axis is fixed in the observer's space and is positively directed away from the bat. By v and v', we denote, respectively, the velocities of the ball immediately before and after the brief period of contact. As is shown in Fig. 2, the directions of these velocities are given by angles θ and ϕ. We write no equations covering the work done by the muscles of the batter but designate the sig-

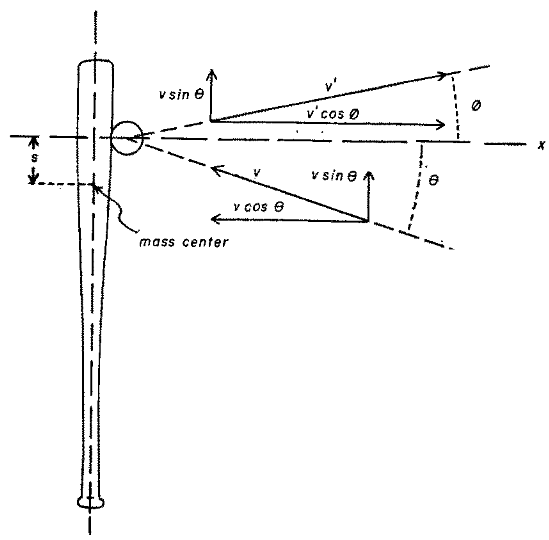

FIG. 2. Illustration of symbols defined and employed in the text.

nificant results of that work by the symbols defined below.

$V = x$ component of the velocity of the mass center of the bat at contact.

ω = angular velocity of the bat at contact with respect to a vertical axis of rotation passing through its mass center.

s = distance of the contact point on the bat from the right section containing the mass center. This quantity is positive when the ball strikes between the mass center and the outer end of the bat.

$V' = x$ component of the velocity of the mass center of the bat as the bat and ball separate.

During the period of contact, the bat expends momentum upon the ball while drawing an additional supply from the continued muscular exertion of the batter, who communicates to it during that period a linear impulse K in the x direction and an angular impulse Ω about the axis of ω. By equating linear impulse to the momentum gained by bat and ball, we obtain

$$K = M(V' - V) + m(v'\cos\phi - v\cos\theta). \quad (1)$$

Treating angular momentum in the same way yields

$$\Omega = ms(v'\cos\phi - v\cos\theta) + I(\omega' - \omega), \quad (2)$$

in which ω' is the angular velocity of the bat as the ball leaves it. We define two further quantities:

$u = x$-directed velocity of the contact point of the bat when the ball first touches it, and
$u' = x$-directed velocity of the same point as the ball departs.

By definition of the coefficient of restitution we have

$$e(u - v\cos\theta) = v'\cos\phi - u', \quad (3)$$

and by addition of collinear velocities

$$u = V + \omega s \quad \text{and} \quad u' = V' + \omega' s. \quad (4)$$

These five equations suffice for the determination of the four primed unknowns. In particular, we find for the departing ball

$$v'\cos\phi = \frac{(1+e)(V+\omega s) + v\cos\theta[(m/M) + (ms^2/I) - e] + (K/M) + (s\Omega/I)}{1 + (m/M) + (ms^2/I)}. \quad (5)$$

In applying this equation to a pitched and batted ball, it must be remembered that v is a negative quantity, all others usually being positive.

In many cases, certain quantities may be dropped from Eq. (5) with little effect upon v'. If the collision is of brief duration,[1] as it is with hard and elastic materials, the impulses K and Ω may be neglected. When the collision takes place near to the mass center of the bat, as it very frequently does, all terms containing s may be deleted. As simplified by these approximations Eq. (5) becomes

$$-v'/v = [a(1+e) + (e-r)\cos\theta]/(1+r)\cos\phi \quad (6)$$

in which the new symbols $r = m/M$ and $a = -V/v$ first appear. Eq. (6) contains only dimensionless ratios.

SWINGING THE BAT

Before discussing Eq. (5) we consider the process of endowing the bat with the velocities V and ω. In general, it is achieved by jointed links in a manner approximately illustrated by the carpenter's hammer. In the use of this tool the upper arm, the forearm, and the hammer handle—a system of links hinged together and to a stationary torso—cooperate to impart velocity to the hammer head. Power is fed in at three joints by applied torque at shoulder, elbow, and wrist. This analogy, however, is incomplete since the torso of the batter is far from stationary, and its advance and rotation may even provide the major part of the bat velocity V.

Available scientific analyses of effective batting form are few and qualitative. We cite here only the pamphlet *Science of Baseball*, prepared by Iwanami, Inc. of Tokyo under the supervision of Professor Ichiro Tani and Professor Kyoichi Nitta of Tokyo University. This publication, though nonmathematical, is a serious and well-illustrated study which is of little direct use to the American reader since it is entirely in the Japanese language. It confirms and renders quantitative the general image of the batsman's performance which thoughtful athletes and acute observers already entertain. The surge of output power directed to the activation of the bat, particularly in slugging, starts at ground level and

[1] *Science of Baseball*, cited below, states upon photographic evidence that the duration of contact between the bat and a well-hit ball is about 5 msec.

rises in fluent coordination. First comes the step, carrying the whole body toward the approaching sphere. While the step is in progress, the rotation of the hips begins but, naturally, this process cannot proceed forcibly until the advancing spikes find their grip in the soil. At about that instant, the powerful muscles which control waist rotation take up the duty of bringing the bat (and the whole upper torso) around. Thus far the cocked arms and bat have been merely riders, but now their contribution to angular velocity begins, though hip and torso are still in rotation. The forward arm (left, in the case of a right-handed batter) does the greater share of the work in this phase, pulling the bat around while its outer end swings outward, not so much by propulsion from the wrists as by what is often called centrifugal force. Last, the angle between forearms and bat, already obtuse, is further increased both by wrist flexure and by a push–pull action of the two forearms.

This relay of time-overlapping forces and torques does some 300 J of work upon the batter's weapon in a fifth of a second, bringing the trademark across the plate with a speed of some 100 ft/sec. The bat achieves its rendezvous, if all goes well, at a point a few inches in front of the plate where $v=100$ ft/sec. The ball is crushed to half its regulation diameter, recovers, and takes up its return flight at $v'=140$ ft/sec possibly.

The performance described above is of course subject to variation. Some batters assume the swinging stance initially and dispense with the step; some space the hands,[2] and some even bat cross-handed. The relative contributions to ultimate bat energy from the component torques naturally vary with physical build and muscular equipment. What we know is derived from laboratory batting: wholehearted swings at fat pitches. So far as known there are no really informative motion pictures of either pitching or batting under combat conditions.

Nice timing of the swing is required for the production of fair balls; from the Iwanami photographs one deduces that a timing uncertainty of ± 0.01 sec makes the difference between a hit over second base and a foul near the first or third base foul lines. The high frequency of such fouls shows that even experts do not find it simple to place a complex muscular event in time with a deviation of the order of $\frac{1}{5}$ of the athlete's reaction time.

It may be and has been argued that the kinetic energy of the bat (or other club) at contact equals the total external work performed at the several bodily joints and should not be affected by the time distribution of power at any of these sites. Why then should the muscular forces be called out sequentially in a particular program rather than activated simultaneously, which would appear to be the woodchopper's time-tried method? Indeed, one may argue that the continuous swing, with all bodily hinges functioning throughout the stroke, *must* produce the greater external kinetic energy since it wastes less energy internally. This seems to follow from the known fact that the maximum external work performed in the contraction of a muscle varies in some inverse manner with the velocity of contraction. High velocities of contraction are, therefore, to be avoided; but this cannot be done by holding the muscle in reserve until the available time for its exercise is largely past.

The refutation of this line of thought is found partly in the fact that the useful work to be done is all against inertia. The work, for instance, put out by the wrists, is proportioned to the resisting force of inertial reaction, and this is called out only by high acceleration, but high acceleration of this kind continued throughout the full duration of the swing would give the bat too much angular displacement, causing it, in the case of a right-handed batter, to point around toward first base. Imparting high velocity to the mass center is only one of the desiderata: this velocity must be attained with the implement in a certain place with a certain orientation. So each of the component torques should be turned on at an instant preceding the instant of contact by just time enough to permit the associated degree of freedom to be exercised throughout its maximum possible extent under maximum muscular exertion.

A more compelling consideration affecting optimum batting form is the fact that the baseball batter must select his procedures under

[2] This form trades off flexible-wrist articulation for more positive bat control. It is a minority choice, particularly in the present era of home-run emphasis, but at least two Hall-of-Fame batters, Wagner and Cobb, elected it.

severe time pressure. The pitched ball, in either the hard or soft form of the game, spends about $\frac{2}{3}$ of a second in its flight to the plate. In this moment the batter must make his observations, complete his forecast of the manner of the ball's arrival, decide upon his consequent plan of action, and get full instructions to his muscles with enough time left to allow them to give the bat suitable values for its thirteen coordinates. Incidentally, these things must be done in hostile surroundings, at appreciable personal risk, under an intense feeling of individual responsibility, and often subject to high-level acoustic annoyances. By comparison, the problems of the tournament golfer at the tee or on the green under conditions of leisure and acoustic sanctity appear simple indeed.[3]

Naturally the batter wishes to postpone his commitment as long as possible so as to have the benefit of the latest and most determinative observations. Every baseball player is so thoroughly familiar with the standard parabolic trajectory that a good look at a few feet of such a flight enables him to extrapolate it to earth, but pitched balls do not always follow such a path and each one is a new ballistic problem to the batter. So he puts off his decision until there is barely enough time left to carry it out. For him the best method of swinging a bat is along the path of minimum time. Being ignorant of the relevant neuromuscular constants of any batter we may not calculate such a path, but we readily see the possible advantage of deferring the wrist break, since an early completion of the work at this joint would increase the moment of inertia of the compound linkage rotating about the shoulder joints or body axis. Batters with strong hands and wrists—and all distinguished sluggers seem to be so equipped—are certainly able to complete the work at the wrist joints in less than the time required to bring the arms about. The baseball batter has another good reason for holding the wrist joints in the cocked position as long as possible: it gives him more time to look over the pitch. If the decision is difficult he may and often does commence the arm swing while continuing his deliberations. If, at the last moment, he decides against a whole-hearted swing no harm has been done him, for the umpire does not regard his effort as a strike unless there has been a break at the wrists, no matter how much bodily rotation and threat of action has been manifested.[4]

[3] Opinions differ. One athlete, uniquely qualified by experience in both tournament golf and World Series baseball, found the pressure more severe in the former activity.

[4] This assurance does not hold in the case of an unsuccessful bunt attempt.

COLLISION QUESTIONS

We return to Eqs. (5) and (6) and an inspection of the conditions affecting the velocity of the batted ball. In general, the faster the pitch the faster is the ball's departure, but this is not invariably the case. Evidently the statement should be reversed in the case of a massive and relatively inelastic ball or a ball struck with a light bat at a point far from the mass center. The best technique for hitting an indoor or playground ball which has been softened up by use is different from that which is most effective on a hard ball having a high value of e. In the latter case the duration of contact is short and the impulses K and Ω are, therefore, small. When collision has been initiated there is little more for the batter to do. But with a soft and inelastic ball (low e) the duration of contact may be great enough to make continued application of force to the bat produce important increments of momentum. The softball batter should in no sense throw his bat at the ball but should follow through forcibly.

At what point along the bat should the ball be caused to strike? The answer depends upon whether the batter is interested in giving high velocity to the ball or in protecting his hands from hinge reaction, as many mechanics books seem to assume. Hinge reaction is the lateral force which may be transmitted to the hands from the impact of the ball. There is nothing unpleasant about it, as one may easily demonstrate with the relatively heavy softball. Hinge reaction does not sting the hands. The sting which is sometimes experienced results from the impact of the ball upon an antinode of the bat's transverse vibration. Such impacts should be avoided because they are unpleasant, because they sometimes break bats, and because they divert useful linear kinetic energy into vibrational motion. Of course, Eqs. (5) and (6) do not apply to such collisions.

The plane motion of the bat just after the collision is given by V' and ω'. We set up a coordinate y to express distances of points along the bat axis from the mass center, choosing the positive direction of y as toward the handle. Then the linear velocity in the x direction for any point on the axis with coordinate y is $V-\omega y$ before the collision, and $V'-\omega' y$ immediately thereafter. By equating these expressions we obtain $y=(V-V')/(\omega-\omega')$, which locates the point where the motion is undisturbed by the collision. By inserting the expressions for V' and ω' which one obtains from Eqs. (1) to (4) we find the undisturbed point to be at $y=k^2/s$. Of course s may be chosen, speaking theoretically, so as to place the undisturbed point at the batter's hands if desired.

It is likely that the batter will prefer to meet the ball at a point yielding a high value of v', but there is little opportunity here for advantageous choice. The terms in Eq. (5) containing s^2 produce an effect which is smoothly symmetrical about the mass center and cannot therefore be of significance for small (positive or negative) values of s. Of the linear s terms, the one involving Ω is inappreciable in hardball situations; the term $s\omega$ does indicate the desirability of meeting the ball at a point outside the mass center, since angular velocity is always present. It would be of interest to observe the values of s utilized by eminent long-ball hitters. Under the dubious assumption that the batter is able to impart to the bat a certain constant amount of kinetic energy (translational plus rotational) it may be shown from Eq. (5) that the velocity of the departing ball has a true maximum when $s=I\omega/MV$.

What are the effects of the mass of the bat, and how should the best mass be selected? If we assume the collision to be essentially instantaneous and hence neglect the impulses K and Ω, then Eq. (6) shows the velocity of the batted ball to be

$$v'=\{V(1+e)+|v|(e-r)\cos\theta\}/(1+r)\cos\phi.$$

Evidently this velocity may be increased by decreasing $r(=m/M)$, that is to say, by using a massive bat, but since r is additively associated with terms of the order of unity the returns diminish after the bat's mass has been increased to a few times that of the ball. Furthermore, as M increases, the velocity V which can be imparted to the bat must decrease, and with it the velocity of the departing ball. At least one may make the qualitative statement that the mass of the bat should be large in comparison to that of the ball yet small in relation to the batter's arms.[5] Such a compromise is possible and is uniformly employed, except that the child batters in Little Leagues often struggle with implements which, for them, are unreasonably massive.

A rough theory of optimum bat mass may be constructed on the assumption that that bat is best which requires the least energy input to impart a given velocity to the ball. This defines and applies a kind of efficiency criterion to bat mass. Rearrangement of Eq. (6) gives the precollision bat velocity as

$$V=[(1+r)v'\cos\phi+(e-r)v\cos\theta]/(1+e). \quad (7)$$

The kinetic energy of the bat, neglecting now the relatively small angular kinetic energy, is $W=MV^2/2$ or, by virtue of Eq. (7),

$$W=[M/2(1+e)^2][(1+r)v'\cos\phi+(e-r)v\cos\theta]^2.$$

It is found that this expression for W has a minimum when

$$r=(v'\cos\phi+ev\cos\theta)/(v'\cos\phi-v\cos\theta)$$
$$\cong(v'+ev)/(v'-v). \quad (8)$$

If the batted ball is to depart with just the reverse of its approach velocity we have $r=(1-e)/2$. This rather typical case is selected for examination because of its arithmetical simplicity. The value of e for baseballs has been measured at the National Bureau of Standards[6]

[5] There is on record (*This Week* magazine, 20 May 1962) a faintly scientific experiment intended to show something about the effect of bat mass upon ball velocity. In this test the distinguished batsman, Roger Maris, batted for distance with 5 different new bats whose weights varied from 33 to 47 oz. These bats, incidentally, were stated by the manufacturer to be reproductions of the favored implements of some of history's greatest home-run hitters. The pitching was by a veteran batting-practice pitcher of the New York Yankees, who attempted to deliver hitable pitches of uniform quality. Maris batted out five long fly balls with each bat and the twenty-five ranges were measured. The correlation coefficient of range with bat mass is found from the published data to be 0.41, indicating a substantial positive relationship. Maris' own favored bat was the lightest of the set, and with it the smallest of the five mean ranges was achieved. This batter, strong enough to wield the heavy, long-range bats, nevertheless prefers a light and maneuverable implement for dealing with unfriendly pitching.

[6] National Bureau of Standards Research Paper RP1624.

where official league balls of the year 1943 were found to have the mean value $e=0.41$. For the case considered this gives $r=0.29$ or $M=3.4\,m$.

It is known that approximations in this theory have had the effect of producing a low value of M, but even the complete elimination of the approximations would not get the theoretical bat mass up to the observed values, which are around $M=7\,m$ in baseball, and $M=5\,m$ in softball. Apparently the principle of least work is not fully binding upon athletes who are much more interested in effectiveness than in efficiency.

The design of bats is presently more traditional than rational, a fact not generally acknowledged or even realized. We read as follows[7]: "Today's bats are scientifically designed and standardized in various sizes, every one shaped and balanced accurately to suit the size, strength, and hitting style of the player."

The credulous present writer followed up this announcement eagerly, asking for references to the researches that had made all this possible and for formulas relating bat constants to measurable human properties. The entire claim was immediately cut down to a statement that a considerable variety of bat sizes and masses is available to the public choice. Bats are not yet sold by prescription.

In the quest for the optimum bat, two opposed desiderata must be compromised. The mass should be zero for maximum wieldiness, yet large (relatively to the ball mass) in order to predominate in the momentum exchange. What one might do is to place the mass where it will do the most good, and the familiar shape of the bat shows some progress in this direction, though not nearly so much as is evident in the golf driver. The mass should be where the collision is to occur. The so-called "bottle bat" once affected by sluggers was a step in just the wrong direction. Of course, batting with a golf driver or other implement with highly concentrated mass would impose upon the already overburdened batter the necessity for precise control of the y coordinate, yet a better concentration than one sees now could surely be achieved by hollow construction of the bat handle and/or use of materials other than the traditional woods.

[7] "Baseball Instructor's Guide," The Athletic Institute, Chicago, Illinois.

Before expending effort upon the improvement of bats the physicist may well ask himself the question, "Why?" Perhaps it is not obvious that the game of baseball would be improved by a scientific disturbance of the existing balance of power between pitcher and batter. There is reason, however, to think that it would. The pitchers' battle is a spectacle more admired than enjoyed. Both spectators and athletes might favor greater general participation, feeling that there is something wrong with a game in which a player may go through an entire contest without once participating in the defensive effort. On the commercial side it is well known that home runs sell tickets.

At least one manufacturer markets bats labeled "Flexible Whip Action." Flexible whip action has aided golfers and pole vaulters and it might help batters. Whip action requires that the acceleration imposed upon the bat by the user's grip on the handle be so great that the massive outer end fails to keep up and bends the implement into a curve, convex on the leading side. The hope is that the energy thus stored is returned at about the instant of collision, but no bats yet introduced are flexible enough to afford much storage. A whip-action bat with which the writer experimented gave indication that its outer end might lag about 2 mm during a high-torque swing.

BATTING SPECIALTIES

The fungo bat, a tool for knocking baseballs about in practice sessions, is a lighter stick than any which would be used on the same ball in competition. Its low mass ($M=4.5\,m$, approximately) is rational in view of the fact that the fungo bat is applied to a ball which has practically zero velocity, a situation in which Eq. (8) prescribes the even lower value $M=m$. The development of a light bat for this specialty was certainly not guided by theory but was perhaps influenced by the fact that a part of the manipulation must be performed with one hand while the user tosses up the ball with the other. Also, this bat is never used in the production of home-runs, where mass has its advantages, but in an activity requiring good control of the ball's velocity vector.

The complementary case, in which the ball has initial velocity but the bat little or none, is bunting. When the bat is presented at rest to receive the ball at the mass center Eq. (6) takes the form $-v'/v=(e-r)\cos\theta/(1+r)\cos\phi$. Inserting the known constants of bats and balls, and assuming $\theta=\phi$, we find that the hard ball loses about 80% of its speed in the bunting collision and the soft ball more than 90%. These estimated loss percentages are on the high side since in the usual execution of a bunt the collision is not completely ballistic. The bat is held in a firm manner which causes it to transfer momentum to the batter and endow the ball with a corresponding additional positive momentum increment. However, this effect is small because of the soft coupling between bat and batter.

The essence of success in bunting is nice placement, which requires better control of v' than is ever thought of in straight-away hitting. If the process described above should seem likely to send the ball too strongly forward the rebound may be reduced in any amount by making the bat velocity negative, a maneuver commonly observed with the baseball but seldom with the softball. For the drop-dead type of bunt we have from Eq. (6), $V=(e-r)v\cos\theta/(1+e)$. Since v is essentially negative, the bat velocity V must also be negative so long as $e>r$, a condition always satisfied except with the lightest of bats and deadest of balls.

DIRECTED BATTING

Contrary to published how-to-bat instructions which might be cited, the ordinary law of reflection does not correctly predict the direction to be taken by the batted ball. This law stands in need of corrections for the effects of spin, friction, imperfect elasticity, yielding of the bat under impact, and bat velocity in the observer's frame of reference. With respect to the last correction, angles of incidence and reflection are equal only in a coordinate system moving with the velocity possessed by the contact spot of the bat at the instant of collision. When these angles are equal in such a moving coordinate system—the bat velocity being positive—the angle of incidence exceeds the angle of reflection if these angles are measured in a frame of reference fixed to the earth. It is for this reason that the distinction between θ and ϕ was retained in the foregoing formulas. Bat and ball velocities are such that this effect alone might cause θ to be twice as large as ϕ if other effects than the motion of the axes did not play compensating roles.

Actually, both the imperfect elasticity of the ball and the transfer of momentum from ball to bat tend to increase the angle of reflection and restore an appearance of validity to the simple reflection law. With all effects operating it is impossible to cover the situation with any simple statement. In bunting, the angle of incidence is the lesser of the two since the reflector is at rest in the observer's space, but in the case of a long drive down a foul line the angle of incidence may be the greater by 25%.

To support these assertions it is economical to write Eq. (6) in the form

$$-v'/v=(D+\cos\theta)/E\cos\phi, \qquad (9)$$

in which the meaning of D and E is evident from a glance at Eq. (6). Now ϕ is expressed by $\tan\phi=-v\sin\theta/v'\cos\phi$, as is clear from Fig. 1. For a ball batted in any given direction the sum of θ and ϕ is, of course, known; in the case of a drive down a base line, for eaxmple, we have $\theta+\phi=\pi/4$. The three relations just given suffice for the determination of v'/v and the two angles, on the basis of given or assumed values of r, a, and e. The speed imparted to the ball by a given blow is a function of the angles involved in the collision. By the methods just stated, it may be shown that a drive down a foul line may have over 5% greater velocity than would have been the case with the swing of the bat timed to send the ball over second base. This much added velocity means about 10% more range.

It is unlikely that any batter can be either improved or misled by equations of mechanics except insofar as such equations may lead to revision of the conditions within which he is free to strive. The achievements of the athlete are confined within limits which he has no power to modify. These boundary conditions are established by rules committees, makers of equipment, and the instructors who prescribe athletic form and technique. Scientific descriptions of what the athlete is doing are likely to have for him an interest which is merely academic, in the common, contemptuous sense of that abused word.